A Parallel Four Step Domain Decomposition Scheme for Coupled Forward Backward Stochastic Differential Equations

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Abstract

Motivated by the idea of imposing paralleling computing on solving stochastic differential equations (SDEs), we introduce a new Domain Decomposition Scheme to solve forward-backward stochastic differential equations (FBSDEs) parallely. We reconstruct the Four Step Scheme in [23] with some different conditions and then associate it with the idea of Domain Decomposition Methods. We also introduce a new technique to prove the convergence of Domain Decomposition Methods for systems of quasilinear parabolic equations and use it to prove the convergence of our scheme for the FBSDEs.

1   Introduction

The theory of forward-backward stochastic differential equations (FBS-DEs) is a very active field of research since the first work of Pardoux and Peng [28] and Antonelli [1] came out in the early 1990s. These equations appear in a large number of application fields in stochastic control and financial mathematics. We refer to the monograph [11], [29] for details, further development and applications. Such systems strongly couple a forward stochastic

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differential equation with a backward one; and they can be written as a kind of stochastic two-point boundary value problems

$$\begin{cases} dX_t = b(t, X_t, Y_t)dt + \sigma(t, X_t, Y_t)dW_t, \\ dY_t = -\hat{b}(t, X_t, Y_t)dt - \hat{\sigma}(t, X_t, Y_t)dW_t, \\ X_0 = x, Y_T = g(X_T). \end{cases} \tag{1.1}$$

Together with the theoretical studies on the systems (see [1], [7], [8], [6], [23], [25], [26], [28]), finding an efficient numerical scheme for FBSDEs has also become an important part of the theory. In order to solve a system of FBSDEs, we need to use the "decoupling PDE" technique, based on the so-called four step scheme (see [23], [25], [26]). In which, the system of FBSDEs is associated with a quasilinear parabolic system of the following type

$$\begin{cases} \frac{\partial \theta}{\partial t} + \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2 \theta}{\partial x_i \partial x_j} + < \nabla \theta, b(t, x, \theta) > + \hat{b}(t, x, \theta) = 0, \text{ in } (0, T) \times \mathbb{R}^n, \\ \theta(T, x) = g(x), \text{ on } \mathbb{R}^n, \end{cases}$$
$$\tag{1.2}$$

where $\theta(t, x)$ is a vector of m components $\theta = (\theta^1, \ldots, \theta^m)$, $m \in \mathbb{N}$.
From here, there are two directions to solve FBSDEs. The first trend is to solve FBSDEs by using the decoupling technique combining with some probability methods to avoid treating the PDEs directly (see [3], [5], [8], [27]). The second trend is to solve directly the PDEs. The first paper in this direction is the one of Douglas, Ma, Protter [10], in which the PDE is treated by a finite difference method. Later in 2008, Ma, Shen and Zhao proposed a new approach based on the Hermite-spectral Method to treat the PDE (see [24]), which is then proved to be much more better than the previous one. In this paper, we present a new approach, still based on the second trend, to the coupled FDSDEs problem, by combining the classical Four Step Scheme with Domain Decomposition Methods or Schwarz Methods, with Waveform Relaxation. The idea is to impose Parallel Computing on solving SDEs numerically. We reconstruct the Four Step Scheme with some new conditions and then associate it with Schwarz Waveform Relaxation Methods to parallelize the system of quasilinear parabolic equations (1.2): System (1.2) is divided into I subproblems, and each problem is solved seperatedly. The scheme is then proved to be well-posed and stable. Up to what we know, this is the first attempt trying to apply Domain Decomposition Algorithms

to stochastic differential equations.

In the pioneer work [17], [18], [19], P. L. Lions laid the foundations of the modern theory of Schwarz Algorithms. With the development of parallel computers, the interest in Schwarz Methods have grown rapidly, as these methods lead to inherently parallel algorithms. However, the problem of convergence of Schwarz Methods still remains an open problem up to now. In his pioneer work [17], [18], [19], P. L. Lions has proved that the classical Schwarz Method for Linear Laplace Equation is in fact equivalent to a sequence of projections in a Hilbert space. Moreover, he also observed that the Schwarz Sequences of linear elliptic equations is related to Minimum Methods over product spaces. This observation was used later by L. Badea in [2] to prove the convergence of the classical Schwarz Method for a class of linear elliptic equations.

Later, in [14] and [15], M. Gander-A. Stuart and E. Giladi-H. B. Keller applied Schwarz Methods to the 1-dimensional linear advection-diffusion equation. Refering to the paper [4], they call Schwarz Methods applied to parabolic equations by Schwarz Waveform Relaxation Algorithms. The techniques of proving the convergence used in these papers were Laplace and Fourier Transforms and some explicit calculations. An extension to the nonlinear reaction-diffusion equation in 1-dimension was considered in [13]. With the hypothese $f'(c) \leq C$ in [13], proofs of linear convergence on unbounded time domains, and superlinear convergence on finite time intervals were then given in case of n subdomains, based on some explicit computations on the linearized equations. Another extension to monotone nonlinear PDEs in higher dimension was considered by Lui in [20], [21], [22]. The main idea of the papers is based on the well-known Sub-Super Solutions Method in the theory of partial differential equations and the initial guesses are usually sub or super solutions of the equations. Recently, an extension to Systems of Semilinear Reaction-Diffusion Equations was investigated in [9]. This is the first paper trying to apply Schwarz Methods to a system of PDEs in 1-dimension and the proof of convergence is based strongly on the technique introduced in [13].

In order to solve FBSDEs by Schwarz Methods, we encounter the system of quasilinear parabolic equations (1.2) in n-dimension. We then introduce a new technique, which allows us to study the convergence of Schwarz Algorithms for systems of nonlinear equations in n-dimension.

2 Forward-backward stochastic differential equations

The structure of this section is as follows: In Section 2.1, we will give the definition of forward-backward stochastic differential equations, then state some results on the existence and uniqueness of the equations; these results will be proved in Section 2.2.

2.1 Existence and uniqueness results

Let $\{W_t : t \geq 0\}$ be a d-dimensional Brownian motion defined on the probability space $(\Omega, \mathfrak{F}, P)$. We define by $\{\mathfrak{F}_t\}$ the σ-field generated by W. We suppose that $\{\mathfrak{F}_t\}$ contains all the null sets of $\mathfrak{F}$ and consider the following forward-backward SDEs

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s, \\ Y_t = g(X_T) + \int_t^T \hat{b}(s, X_s, Y_s)ds + \int_t^T \hat{\sigma}(s, X_s, Y_s, Z_s)dW_s, \end{cases} \tag{2.1}$$

where t belongs to $[0, T]$; the processes X, Y, Z take values in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^{m \times d}$, respectively and b, $\hat{b}$, σ, $\hat{\sigma}$, g take values in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^{n \times d}$, $\mathbb{R}^{m \times d}$ and $\mathbb{R}^m$, respectively.

Since we are only looking for *ordinary adapted solutions* of the FBSDEs (2.1) (i.e. solutions which are $\{\mathfrak{F}_t\}$-adapted and square-integrable, and satisfy (2.1) P-almost surely), we can write (2.1) in the following form

$$\begin{cases} dX_t = b(t, X_t, Y_t)dt + \sigma(t, X_t, Y_t)dW_t, \\ dY_t = -\hat{b}(t, X_t, Y_t)dt - \hat{\sigma}(t, X_t, Y_t, Z_t)dW_t, \\ X_0 = x, Y_T = g(X_T). \end{cases} \tag{2.2}$$

Now we state the conditions that we impose on (2.1) and (2.2):

(A1) The functions b, $\hat{b}$, σ, $\hat{\sigma}$, g are C^1-functions with bounded partial derivaties; and g is bounded in $C^{2+\delta}(\mathbb{R}^m)$ for some δ in $(0, 1)$.

(A2) The matrix σ satisfies

$$|\sigma(t, x, y)| \leq C,$$

and

$$\sigma(t, x, y)\sigma^T(t, x, y) \geq \nu(|y|)I, \forall (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m,$$

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where ν is a positive continuous function, and C is a positive constant.
(A3) We impose the following assumptions on $\hat{\sigma}$:
There exists a positive continuous function κ such that

$$\sup\{ |z| : \ \hat{\sigma}(t, x, y, z) = 0\} \leq \kappa(|y|), \forall (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m. \quad (2.3)$$

For fixed (t, x, y), the function $z \mapsto \hat{\sigma}(t, x, y, z)$ is bijective, $\hat{\sigma}^{-1}(t, x, y)(\zeta)$ is contiuous with respect to t, x, y and ζ; and there exists a continuous function λ from $\mathbb{R}$ to $\mathbb{R}_+$, such that

$$|\hat{\sigma}^{-1}(t, x, y)(\zeta) - \hat{\sigma}^{-1}(t, x, y)(\zeta')| \leq \lambda(|y|)|\zeta - \zeta'|^\alpha, \forall \zeta, \zeta' \in \mathbb{R}^{m \times n}, \quad (2.4)$$

where α is a constant lying in $[1, 2)$.
(A4) There exists a positive function η and a positive constant C such that for all (t, x, y) in $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m$,

$$|b(t, x, y)| \leq \eta(|y|)$$

and

$$\hat{b}(t, x, 0) \leq C.$$

(A5) We suppose also that for all $k \in \{1, \ldots, m\}$ and for all $(t, x, y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_m)$ in $[0, T] \times \mathbb{R}^n \times \mathbb{R}^{m-1}$:

$$\hat{b}^k(t, x, y_1, \ldots, y_{k-1}, 0, y_{k+1}, \ldots, y_m) = 0,$$

and $\hat{b}^k(t, x, y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_m)$ is decreasing in y_k.

Assuming that Y_t takes the form $\theta(t, X_t)$, P-almost surely, for all t in $[0, T]$, by the Itô's formula, we can transform the backward SDE in (2.2) into the following system of PDEs

$$\begin{cases} \frac{\partial \theta}{\partial t} + \sum_{i,j=1}^n a_{i,j} \frac{\partial^2 \theta}{\partial x_i \partial x_j} + < \nabla \theta, b(t, x, \theta) > +\hat{b}(t, x, \theta) = 0, \ \text{in} \ (0, T) \times \mathbb{R}^n, \\ \theta(T, x) = g(x), \ \text{on} \ \mathbb{R}^n, \end{cases}$$

$$(2.5)$$

where we define $\sigma^T(t, x, \theta)$ to be the transposed matrix of $\sigma(t, x, \theta)$ and $(a_{i,j})$ to be $\frac{1}{2}\sigma(t, x, \theta)\sigma^T(t, x, \theta)$. The result on the existence and uniqueness of solutions of (2.2) then follows

Theorem 2.1. *(Existence and Uniqueness Theorem) Suppose that Conditions $(A1)$- $(A4)$ above are satified, then Equation (2.2) admits a unique solution (X, Y, Z) defined as follows:*

The process X is the solution of the following forward SDE

$$X_t = x + \int_0^t b(s, X_s, \theta(s, X_s))ds + \int_0^t \sigma(s, X_s, \theta(s, X_s))ds, \qquad (2.6)$$

where θ is the unique solution of (2.5).

The processes Y_t, Z_t are then $\theta(t, X_t)$ and $z(t, X_t, \theta(t, X_t), \nabla\theta(t, X_t))$, where z is a smooth mapping from $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n}$ to $\mathbb{R}^{m \times d}$ satisfying

$$\xi\sigma(t, x, y) + \hat{\sigma}(t, x, y, z(t, x, y, \xi)) = 0, \forall(t, x, y, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n}. \tag{2.7}$$

Remark 2.1. *Assumptions $(A1)$ - $(A4)$ is similar to Assumptions $(A1)$ - $(A4)$ in [23]. However, in (2.4), α can vary in $[1, 2)$ while in the condition (2.12) of $(A3)$ [23], $\alpha = 1$. $(A5)$ will be used later in the proof of convergence for the Parallel Four Steps Domain Decomposition Scheme.*

2.2 Proof of existence and uniqueness results

This subsection is devoted to the proof of the existence and uniqueness results stated in Section 2.1. We first consider the algebraic equation (2.7). The following result shows that (2.7) has a solution.

Proposition 2.1. *Under Assumption $(A3)$, Equation (2.7) has a unique continuous solution z, that satisfies the estimate*

$$|z(t, x, y, \xi)| \le \lambda(|y|)|\xi||\sigma(t, x, y)|^\alpha + \kappa(|y|), \quad \forall(t, x, y, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n}.$$

Proof. Since $z(t, x, y, \xi)$ is equal to $\hat{\sigma}^{-1}(t, x, y)(\xi\sigma(t, x, y))$ and $\hat{\sigma}^{-1}$, σ are continuous, then $z(t, x, y, \xi)$ is continuous. Finally, Conditions (2.3) and (2.4) give

$$
\begin{aligned}
|z(t, x, y, \xi)| &\le |\hat{\sigma}^{-1}(t, x, y)(\xi\sigma(t, x, y)) - \hat{\sigma}^{-1}(t, x, y)(0)| + |\hat{\sigma}^{-1}(t, x, y)(0)| \\
&\le \lambda(|y|)|\xi\sigma(t, x, y)|^\alpha + |\kappa(|y|)|.
\end{aligned}
$$

$\square$

Now, the following Proposition states a result on the existence and uniqueness of solutions to (2.5):

Proposition 2.2. *Suppose that* $(A1) - (A4)$ *hold. Then the system* (2.5) *admits a unique classical solution* $\theta(t, x)$, *such that* $\theta(t, x)$, $\frac{\partial}{\partial t}\theta(t, x)$, $\nabla\theta(t, x)$, $\Delta\theta(t, x)$ *are bounded in* $C((0, T) \times \mathbb{R}^n)$.

Proof. We first recall a useful result. Let ω be a bounded and smooth enough domain of $\mathbb{R}^n$, we consider the system

$$
\begin{cases}
\frac{\partial\phi}{\partial t} + \sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2\phi}{\partial x_i \partial x_j} + <\nabla\phi, b(t, x, \phi)> + \hat{b}(t, x, \phi) = 0, \text{ in } (0, T) \times \omega, \\
\phi(t, x) = g(t, x), \text{ on } [0, T] \times \partial\omega, \\
\phi(T, x) = g(T, x), \text{ on } \omega.
\end{cases}
$$

$$(2.8)$$

Similarly as in Theorem 7.1, Chapter VII of [16] and Lemma 3.2 of [23], we have the following Lemma

Lemma 2.1. *Suppose that all the functions* a_{ij}, b_i, $\hat{b}$ *are smooth,* g *is bounded in* $C^{1+\delta, 2+\delta}([0, T] \times \omega)$ *with* δ *belongs to* $(0, 1)$; *and for all* $(t, x, y) \in [0, t] \times \mathbb{R}^n \times \mathbb{R}^m$, *we have*

$$\nu_1(|y|)I \leq (a_{ij}(t, x, y)) \leq \nu_2(|y|)I, \tag{2.9}$$

$$|b(t, x, y)| \leq \mu(|y|), \tag{2.10}$$

$$\left|\frac{\partial}{\partial x_l}a_{ij}(t, x, y)\right| + \left|\frac{\partial}{\partial y_k}a_{ij}(t, x, y)\right| \leq \mu(|y|), \tag{2.11}$$

for some continuous positive functions $\nu_1(.)$, $\nu_2(.)$, $\mu(.)$; *and*

$$|\hat{b}(t, x, y)| \leq C_1(1 + |y|), \tag{2.12}$$

$$|<\hat{b}(t, x, y), y>| \leq C_2(1 + |y|^2), \tag{2.13}$$

for some positive constants C_1, C_2. *Then* (2.8) *admits a unique classical solution* $\phi(t, x)$. *Moreover,* $\phi(t, x)$, $\frac{\partial}{\partial t}\phi(t, x)$, $\nabla\phi(t, x)$, $\Delta\phi(t, x)$ *are bounded in* $C((0, T) \times \mathbb{R}^n)$ *by a constant which does not depend on* ω, *and there exists a positive number* δ' *in* $(0, 1)$ *such that* ϕ *belongs to* $C^{1+\delta', 2+\delta'}((0, T) \times \omega)$.

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Now, we will apply this Lemma to our case. First of all, we verify that the conditions of Lemma 2.1 hold. We can see that (2.9) is a consequence of $(A2)$, (2.10) is a consequence of $(A4)$ and (2.11) is a consequence of $(A1)$. We only need to prove (2.12) and (2.13).
Conditions $(A1)$ and $(A4)$ infer that

$$|\hat{b}(t,x,y)| \leq |\hat{b}(t,x,y) - \hat{b}(t,x,0)| + |\hat{b}(t,x,0)| \leq C_1(1 + |y|),$$

and this implies

$$| < \hat{b}(t,x,y), y > | \leq C'_1(1 + |y|)|y| \leq C_2(1 + |y|^2),$$

where C'_1 is a positive constant. Lemma 2.1 then implies that there exists a solution $\phi(t,x)$ for all ω in $\mathbb{R}^n$ bounded and smooth enough.

By a convergence argument similar as in [23], we deduce that (2.5) admits a unique classical solution $\theta(t,x)$, such that $\theta(t,x)$, $\frac{\partial}{\partial t}\theta(t,x)$, $\nabla\theta(t,x)$, $\Delta\theta(t,x)$ are bounded in $C((0,T) \times \mathbb{R}^n)$. $\qquad\square$

We consider the forward SDE on X from (2.2), with the assumption that Y_t can be written under the form $\theta(t,X_t)$

$$
\begin{cases}
dX_t = b(t,X_t,\theta(t,X_t))dt + \sigma(t,X_t,\theta(t,X_t))dW_t, \\
X_0 = x.
\end{cases}
\tag{2.14}
$$

From the Lipschitz condition $(A1)$, we can conclude that (2.14) has a unique solution X, which belongs to $\mathbb{L}^2(0,T)$. We then have the following proof of Theorem 2.1, based on the previous two Propositions.

Proof. The proof is divided into two steps
Step 1: (X_t, Y_t, Z_t) is a solution of (2.2).
We note that the existence and uniqueness of X_t has already been discussed in the previous paragraph, we only need to prove that (Y_t, Z_t) is a solution of the backward SDE in the system (2.2). It follows from Itô's formula that

$$
\begin{aligned}
dY_t &= \left[\frac{\partial\theta}{\partial t} + \sum_{i=1}^{n}\frac{\partial\theta}{\partial x_i}b^i(x,X_t,Y_t) + \sum_{i,j=1}^{n}\frac{\partial^2\theta}{\partial x_i\partial x_j}a_{ij}\right]dt \qquad (2.15) \\
&\quad + \sum_{i=1}^{n}\frac{\partial\theta}{\partial x_i}\sigma^i(t,X_t,Y_t)dW_t.
\end{aligned}
$$

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Equations (2.15), Proposition 2.1 and the fact that θ is a solution of (2.5) lead to

$$dY_t = -\hat{b}(t, X_t, Y_t)dt - \hat{\sigma}(t, X_t, Y_t, Z_t)dW_t. \qquad (2.16)$$

Hence, (X_t, Y_t, Z_t) is a solution of (2.2).

Step 2: (X_t, Y_t, Z_t) is unique.

Let (X_t^*, Y_t^*, Z_t^*) a solution of (2.2), and set Y_t' to be $\theta(t, X_t^*)$, Z_t' to be $z(t, X_t^*, \theta(t, X_t^*), \nabla\theta(t, X_t^*))$. It suffices to show that $Y_t' = Y_t^*$ and $Z_t' = Z_t^*$.

We first try to get some estimate for the quantity $E(|Y_t' - Y_t^*|^2)$. Itô's formula gives

$$d(Y_t' - Y_t^*) = \left[\frac{\partial\theta}{\partial t} + \sum_{i=1}^{n}\frac{\partial\theta}{\partial x_i}b^i(t, X_t^*, Y_t^*) + \sum_{i,j=1}^{n}a_{ij}\frac{\partial^2\theta}{\partial x_i\partial x_j} + \hat{b}(t, X_t^*, Y_t^*)\right]dt$$

$$+ \left[\sum_{i=1}^{n}\frac{\partial\theta}{\partial x_i}\sigma^t(t, X_t^*, Y_t^*) + \hat{\sigma}(t, X_t^*, Y_t^*, Z_t^*)\right]dW_t.$$

By using Itô's formula for the k-th component of $Y_t' - Y_t^*$, $k \in \{1, \ldots, m\}$, we obtain

$$d(Y_t'^k - Y_t^{*k})^2 \qquad (2.17)$$
$$= \left[\frac{\partial\theta^k}{\partial t} + \sum_{i=1}^{n}\frac{\partial\theta^k}{\partial x_i}b^i(t, X_t^*, Y_t^*) + \sum_{i,j=1}^{n}a_{ij}\frac{\partial^2\theta^k}{\partial x_i\partial x_j} + \hat{b}^k(t, X_t^*, Y_t^*)\right] \times$$
$$\times 2(Y_t'^k - Y_t^{*k})dt +$$
$$+ 2(Y_t'^k - Y_t^{*k})\left[\sum_{i=1}^{n}\frac{\partial\theta^k}{\partial x_i}\sigma^i(t, X_t^*, Y_t^*) + \hat{\sigma}^k(t, X_t^*, Y_t^*, Z_t^*)\right]dW_t$$
$$+ \left[\sum_{i=1}^{n}\frac{\partial\theta^k}{\partial x_i}\sigma^i(t, X_t^*, Y_t^*) + \hat{\sigma}^k(t, X_t^*, Y_t^*, Z_t^*)\right]^2 dt.$$

From the previous equation, the fact that θ is a solution of (2.5) implies

$$d(Y_t'^k - Y_t^{*k})^2 \qquad (2.18)$$
$$= 2(Y_t'^k - Y_t^{*k})\{\sum_{i=1}^{n}\frac{\partial\theta^k}{\partial x_i}[b^i(t, X_t^*, Y_t^*) - b^i(t, X_t^*, Y_t')] +$$

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$$+\sum_{i,j=1}^{n}\frac{\partial^2\theta^k}{\partial x_i\partial x_j}[a_{ij}(x,X_t^*,Y_t^*)-a_{ij}(x,X_t^*,Y_t')]+$$

$$+[\hat{b}^k(t,X_t^*,Y_t^*)-\hat{b}^k(t,X_t^*,Y_t')]\}dt+$$

$$+2(Y_t'^k-Y_t^{*k})\left[\sum_{i=1}^{n}\frac{\partial\theta^k}{\partial x_i}\sigma^i(t,X_t^*,Y_t^*)+\hat{\sigma}^k(t,X_t^*,Y_t^*,Z_t^*)\right]dW_t$$

$$+\left[\sum_{i=1}^{n}\frac{\partial\theta^k}{\partial x_i}\sigma^i(t,X_t^*,Y_t^*)+\hat{\sigma}^k(t,X_t^*,Y_t^*,Z_t^*)\right]^2 dt.$$

Since z is a solution of (2.7), we have that

$$\sum_{i=1}^{n}\frac{\partial\theta^k}{\partial x_i}\sigma^i(t,X_t^*,Y_t^*)+\hat{\sigma}^k(t,X_t^*,Y_t^*,Z_t^*) \qquad (2.19)$$

$$=\sum_{i=1}^{n}\frac{\partial\theta^k}{\partial x_i}[\sigma^i(t,X_t^*,Y_t^*)-\sigma^i(t,X_t^*,Y_t')]+$$

$$+[\hat{\sigma}^k(t,X_t,Y_t^*,Z_t^*)-\hat{\sigma}^k(t,X_t^*,Y_t',Z_t')].$$

Since Y_T^* is equal to Y_T', then $E(|Y_T'^k-Y_T^{*k}|^2)=0$. We can infer from (2.18) and (2.19) that

$$E|Y_t'^k-Y_t^{*k}|^2 \qquad (2.20)$$

$$=-E\int_t^T 2(Y_s^k-Y_s^{*k})\{\sum_{i=1}^{n}\frac{\partial\theta^k}{\partial x_i}[b^i(s,X_s^*,Y_s^*)-b^i(s,X_s^*,Y_s')]+$$

$$+\sum_{i,j=1}^{n}\frac{\partial^2\theta^k}{\partial x_i\partial x_j}[a_{ij}(x,X_s^*,Y_s^*)-a_{ij}(x,X_s^*,Y_s')]+$$

$$+[\hat{b}^k(s,X_s^*,Y_s^*)-\hat{b}^k(s,X_s^*,Y_s')]\}ds+$$

$$-E\int_t^T\{\sum_{i=1}^{n}\frac{\partial\theta^k}{\partial x_i}[\sigma^i(s,X_s^*,Y_s^*)-\sigma^i(s,X_s^*,Y_s')]+$$

$$+[\hat{\sigma}^k(s,X_s^*,Y_s^*,Z_s^*)-\hat{\sigma}^k(s,X_s^*,Y_s',Z_s')]\}^2 ds.$$

Using (A1), we deduce from (2.20) that there exists a positive constant M_1 such that

$$E(|Y_t'^k-Y_t^{*k}|^2)+E\int_t^T[\hat{\sigma}^k(s,X_s^*,Y_s^*,Z_s^*)-\hat{\sigma}^k(t,X_s^*,Y_s^*,Z_s')]^2 ds$$

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$$\leq \quad M_1 E \int_t^T |Y_s'^k - Y_s^{*k}|(|Y_s' - Y_s^*| + |Z_s' - Z_s^*|)ds, \qquad (2.21)$$

which leads to

$$E(|Y_t' - Y_t^*|^2) + E \int_t^T |\hat{\sigma}(s, X_s^*, Y_s^*, Z_s^*) - \hat{\sigma}(s, X_s^*, Y_s^*, Z_s')|^2 ds$$

$$\leq \quad M_2 E \int_t^T |Y_s' - Y_s^*|(|Y_s' - Y_s^*| + |Z_s' - Z_s^*|)ds, \qquad (2.22)$$

where M_2 is a positive constant. Condition (2.4) implies that

$$
\begin{aligned}
|Z_t' - Z_t^*| &= |\hat{\sigma}^{-1}\hat{\sigma}(t, X_t^*, Y_t^*, Z_t^*) - \hat{\sigma}^{-1}\hat{\sigma}(t, X_t^*, Y_t^*, Z_t')| \\
&\leq \lambda(|Y_t|)|\hat{\sigma}(t, X_t^*, Y_t^*, Z_t^*) - \hat{\sigma}(t, X_t^*, Y_t^*, Z_t')|^\alpha \\
&\leq M_3 |\hat{\sigma}(t, X_t^*, Y_t^*, Z_t^*) - \hat{\sigma}(t, X_t^*, Y_t^*, Z_t')|^\alpha,
\end{aligned}
$$

where M_3 is a positive constant, since θ is uniformly bounded .
Therefore

$$E(|Y_t' - Y_t^*|^2) + M_4 E \int_t^T |Z_s' - Z_s^*|^{\frac{2}{\alpha}} ds \qquad (2.23)$$

$$\leq \quad M_2 E \int_t^T |Y_s' - Y_s^*|(|Y_s' - Y_s^*| + |Z_s' - Z_s^*|)ds$$

$$\leq \quad M_2 E \int_t^T |Y_s' - Y_s^*|^2 ds + M_2 E \int_t^T |Y_s' - Y_s^*||Z_s' - Z_s^*| ds$$

$$\leq \quad M_2 E \int_t^T |Y_s' - Y_s^*|^2 ds + \frac{M_5}{\epsilon^{\frac{2}{2-\alpha}}} E \int_t^T |Y_s' - Y_s^*|^{\frac{2}{2-\alpha}} ds$$

$$+ \epsilon^{\frac{2}{\alpha}} M_6 E \int_t^T |Z_s' - Z_s^*|^{\frac{2}{\alpha}} ds.$$

where M_4, M_5, M_6 are positive constants and ϵ is a small constant to be chosen. Notice that the last inequality comes from Young's inequality. Choosing ϵ small enough, we get from (2.23) that

$$E(|Y_t' - Y_t^*|^2) \quad \leq \quad M_7 \int_t^T [E(|Y_s' - Y_s^*|^2) + E(|Y_s' - Y_s^*|^2)^{\frac{1}{2-\alpha}}]ds, (2.24)$$

where M_7 is positive constant.

Secondly, we will prove that $E(|Y_t' - Y_t^*|^2)$ is bounded and then deduce

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that $E(|Y_t' - Y_t^*|^2)$ is equal to 0. Setting $G(t)$ to be $E(|Y_t' - Y_t^*|^2)$, we will establish a *New Gronwall Inequality*, to prove that $G(t)$ is bounded. We have the following inequality on G

$$G(t) \ \leq \ M_7 \int_t^T [G(s) + G(s)^{\frac{1}{2-\alpha}} + 1] ds.$$

Now denoting $M_7 \int_t^T [G(s) + G(s)^{\frac{1}{2-\alpha}} + 1] ds$ by $H(t)$, we deduce that $G(t)$ is less than or equal to $H(t)$; and as a result $H'(t)$ is bounded by $M_7[H(t) + H(t)^{\frac{1}{2-\alpha}} + 1]$. This leads to

$$\frac{H'(t)}{H(t) + H(t)^{\frac{1}{2-\alpha}} + 1} \ \leq \ M_7.$$

We define

$$K(r) = \int_0^r \frac{1}{\rho + \rho^{\frac{1}{2-\alpha}} + 1} d\rho.$$

Since $(K(H(t)))'$ is bounded by M_7, then

$$K(H(t)) \leq M_7 T + K(H(0)) = M_8.$$

In addition, we know that K is increasing, which means that $H(t)$ is less than or equal to $K^{-1}(M_8)$. We then infer that $G(t)$ is bounded by a postive constant M_9. Defining $M_9^{-1} G(t)$ by $P(t)$, we then deduce that $P(t)$ is positive and bounded by 1. This leads to

$$P(t) \ \leq \ M_{10} \int_t^T [P(s) + P(s)^{\frac{1}{2-\alpha}}] ds \leq M_{11} \int_t^T P(s) ds,$$

where M_{10}, M_{11} are positive constants; notice that $1 \leq \alpha < 2$.
By the classical Gronwall's Lemma, $P(t)$ is equal to 0, which implies that Y_t' is coincided with Y_t^*. From this, we infer that Z_t' is coincided with Z_t^*.

Now, since every solution (X_t^*, Y_t^*, Z_t^*) of (2.2) can be written as $(X_t^*, \theta(t, X_t^*), z(t, X_t^*, \theta(t, X_t^*)))$ and since (2.14) has only one solution, we can conclude that (2.2) has a unique solution. $\square$

3 The Parallel Four Step Domain Decomposition Scheme

This section is devoted to the construction of the Parallel Four Step Domain Decomposition Scheme and its proof of well-posedness and stability.

3.1 Definition of the Scheme

We now define a new Parallel Four Step Domain Decomposition Scheme, based on the results obtained in Section 2.

Step 1: Find a smooth mapping z satisfying (2.7).

Step 2: Choose l to be a constant large enough and consider the domain $\mathcal{O}_l = (-l, l)^n$. On $\mathcal{O}_l$, consider the following problem instead of (2.5)

$$
\begin{cases}
\frac{\partial \theta^l}{\partial t} + \sum_{i,j=1}^n a_{i,j} \frac{\partial^2 \theta^l}{\partial x_i \partial x_j} + <\nabla \theta^l, b(t, x, \theta^l)> + \hat{b}(t, x, \theta^l) = 0, \text{ in } (0, T) \times \mathcal{O}_l, \\
\theta^l(t, x) = g(x), \text{ on } (0, T) \times \partial \mathcal{O}_l, \\
\theta^l(T, x) = g(x), \text{ on } \mathcal{O}_l.
\end{cases}
$$

$$(3.1)$$

The same arguments as in the proof of Proposition 2.2 show that (3.1) has a unique classical solution $\theta^l(t, x)$, where $\theta^l(t, x)$, $\frac{\partial}{\partial t}\theta^l(t, x)$, $\nabla \theta^l(t, x)$, $\Delta \theta^l(t, x)$ are bounded. Suppose that $\theta^l(t, x) = g(x)$ on $(0, T) \times (\mathbb{R}^n \backslash \mathcal{O}_l)$, then these arguments also show that

$$
\lim_{l \to \infty} ||\theta^l - \theta||_{(L^\infty((0,T) \times \mathbb{R}^n))^m} = 0.
$$

Step 3: Solve the equation (3.1) iteratively in the following manner

- Divide $\mathcal{O}_l$ into I subdomains

$$
\mathcal{O}_l = \bigcup_{p=1}^I \Omega_p = \bigcup_{p=1}^I (-l, l)^{n-1} \times (a_p, b_p),
$$

where $-l = a_1 < a_2 < b_1 < \cdots < a_I < b_{I-1} < b_I = l$. Denote that $S_i = b_i - a_{i+1}$ for i belongs to $\{1, \ldots, I-1\}$ and $L_i = b_i - a_i$ for i belongs to $\{1, \ldots, I\}$.

- Choose a bounded initial guess θ_0^l in $\mathbb{C}^\infty(\mathbb{R}^n)$ at step 0. Associate each subdomain Ω_p with a function $\theta_{p,0}^l$ such that $\theta_{p,0}^l = \theta_0^l$ on Ω_p.

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- Solve the following p-th subproblem at iteration #q

$$\begin{cases} \frac{\partial \theta^l_{p,q}}{\partial t} + \sum_{i,j=1}^n a_{i,j} \frac{\partial^2 \theta^l_{p,q}}{\partial x_i \partial x_j} + < \nabla \theta^l_{p,q}, b(t,x,\theta^l_{p,q}) > + \hat{b}(t,x,\theta^l_{p,q}) = 0, \text{ in } (0,T) \times \Omega_p, \\ \theta^l_{p,q}(.,.,a_p) = \theta^l_{p-1,q-1}(.,.,a_p), \text{ on } (0,T) \times (-l,l)^{n-1}, \\ \theta^l_{p,q}(.,.,b_p) = \theta^l_{p+1,q-1}(.,.,b_p), \text{ on } (0,T) \times (-l,l)^{n-1}, \\ \theta^l_{p,q}(t,x) = g(t,x), \text{ on } (0,T) \times (\partial \mathcal{O}_l \backslash ((0,T) \times (-l,l)^{n-1} \times (\{a_p\} \cup \{b_p\}))), \\ \theta^l_{p,q}(T,x) = g(T,x), \text{ on } (-l,l)^{n-1} \times (a_p,b_p). \end{cases}$$
$$(3.2)$$

For the extreme subdomain Ω_1 (resp. Ω_I), we consider the boundary condition $\theta^l_{1,q}(t,x,a_1) = g(t,x)$ on the left (resp. $\theta^l_{I,q}(t,x,b_I) = g(t,x)$ on the right) in (3.2).

- Suppose that we stop at the iteration #q while solving (3.2).

The following two theorems insist that Step 2 of the algorithm is well-posed and show that the solutions of the subproblems (3.2) converge to the solution of the main problem (3.1) when q tends to infinity.

Theorem 3.1. *(Well-posedness Theorem) Suppose that $(A1) - (A5)$ hold, then at each iteration #q in each subdomain #p, there exists a unique classical solution $\theta^l_{p,q}(t,x)$ for (3.2), such that $\theta^l_{p,q}(t,x)$, $\frac{\partial}{\partial t}\theta^l_{p,q}(t,x)$, $\nabla \theta^l_{p,q}(t,x)$, $\Delta \theta^l_{p,q}(t,x)$ are bounded and the sequence $\{\theta^l_{p,q}\}_{p \in \{1,...,I\}; q \in \mathbb{N}}$ is uniformly bounded (with respect to p and q) in $C((0,T) \times \mathcal{O}_l)$.*

Theorem 3.2. *Under Assumptions $(A1) - (A5)$, we have the convergence*

$$\lim_{q \to \infty} \sup_{p \in \{1,...,I\}} ||\theta^l_{p,q} - \theta^l||_{(L^\infty((0,T) \times (-l,l)^{n-1} \times (a_p,b_p)))^m} = 0. \qquad (3.3)$$

Step 4: We will continue with the values $\theta^l_{p,q}$, $p \in \{1,\ldots,I\}$ that we have got at the end of step 2.

- Let θ^l_q be a function defined on $[0,T] \times \mathcal{O}_l$ such that $\theta^l_q(t,x) = \theta^l_{p,q}(t,x)$ on $[0,T] \times (\Omega_p \backslash (\Omega_{p-1} \cup \Omega_{p+1}))$ for $p \in \{2,\ldots,I-1\}$, on $[0,T] \times (\Omega_p \backslash \Omega_{p-1})$ for $p = I$, and on $[0,T] \times (\Omega_p \backslash \Omega_{p+1})$ for $p = 1$. We can choose $\theta^l_q(t,x)$ such that it is Lipschitz, differentiable with respect to x and t in $\mathbb{R}^n$ and $\mathbb{R}$ and

$$\lim_{q \to \infty} ||\theta^l_q - \theta^l||_{(L^\infty([0,T] \times \mathcal{O}_l))^m} = 0. \qquad (3.4)$$

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- Use θ_q^l, solve the following forward SDE

$$X_t^q = x + \int_0^t \bar{b}_q(s, X_s^q)ds + \int_0^t \bar{\sigma}_q(s, X_s^q)dW_s, \qquad (3.5)$$

where $\bar{b}_q$ is $b(t, x, \theta_q^l(t, x))$ and $\bar{\sigma}_q(t, x)$ is $\sigma(t, x, \theta_q^l(t, x))$.

Using the same arguments as the ones used for (2.14), we can conclude that (3.5) has a unique solution in $\mathbb{L}^2(0, T)$.

Set $Y_t^{q,l} = \theta_q^l(t, X_t^{q,l})$ and $Z_t^{q,l} = z(t, X_t^{q,l}, \theta_q^l(t, X_t^{q,l}), \nabla \theta_q^l(t, X_t^{q,l}))$. The following Theorem says that the sequence $(X_t^{q,l}, Y_t^{q,l}, Z_t^{q,l})$ converges to (X_t, Y_t, Z_t) as q and l tend to infinity.

Theorem 3.3. *(Convergence Theorem) Suppose that all the assumptions in Section 2.1 hold, then as q and l tend to infinity, $(X_t^{q,l}, Y_t^{q,l}, Z_t^{q,l})$ converges to the solution (X_t, Y_t, Z_t) of (2.1) in the following sense*

$$\lim_{l \to \infty} \lim_{q \to \infty} \int_0^T E(|X_t^{q,l} - X_t|^2)dt = 0,$$

$$\lim_{l \to \infty} \lim_{q \to \infty} \int_0^T E(|Y_t^{q,l} - Y_t|^2)dt = 0,$$

$$\lim_{l \to \infty} \lim_{q \to \infty} \int_0^T E(|Z_t^{q,l} - Z_t|^{\frac{2}{\alpha}})dt = 0.$$

3.2 Proof of Theorem 3.1

First of all, we introduce some useful notations which will be needed for the proof. We set

$$M_0 = \max\{||\theta_0^l||_{C([0,T] \times \mathbb{R}^n)}, ||g||_{C(\mathbb{R}^n)}\},$$

and define $\rho_{p,q}(t, x)$ to be $\theta_{p,q}^l(T - t, x)$ for $p \in \{1, \ldots, I\}$ and $q \in \mathbb{N}$.
We can reformulate Systems (3.2) into the following form

$$\begin{cases} -\frac{\partial \rho_{p,q}}{\partial t} + \sum_{i,j=1}^n a_{i,j} \frac{\partial^2 \rho_{p,q}}{\partial x_i \partial x_j} + <\nabla \rho_{p,q}, b(t, x, \rho_{p,q})> +\hat{b}(t, x, \rho_{p,q}) = 0, \text{ in } (0, T) \times \Omega_p, \\ \rho_{p,q}(., ., a_p) = \rho_{p-1,q-1}(., ., a_p), \text{ on } (0, T) \times (-l, l)^{n-1}, \\ \rho_{p,q}(., ., b_p) = \rho_{p+1,q-1}(., ., b_p), \text{ on } (0, T) \times (-l, l)^{n-1}, \\ \rho_{p,q}(t, x) = g(t, x), \text{ on } (0, T) \times (\partial \mathcal{O}_l \backslash ((-l, l)^{n-1} \times (\{a_p\} \cup \{b_p\}))), \\ \rho_{p,q}(0, x) = g(0, x), \text{ on } (-l, l)^{n-1} \times (a_p, b_p). \end{cases}$$

$$(3.6)$$

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One can see that (3.6) are parabolic systems with the initial condition g.

Now, we will prove the Theorem by induction.

At step #1, and in the p-th subdomain, using the same argument as in Theorem 2.1, we can prove that (3.2) admits a unique classical solution $\theta_{p,1}^l(t,x)$, where $\theta_{p,1}^l(t,x)$, $\frac{\partial}{\partial t}\theta_{p,1}^l(t,x)$, $\nabla\theta_{p,1}^l(t,x)$, $\Delta\theta_{p,1}^l(t,x)$ are bounded. Consider the following k-th equation of (3.6), for k in $\{1,\ldots,m\}$

$$
\begin{cases}
-\frac{\partial\rho_{p,1}^k}{\partial t} + \sum_{i,j=1}^n a_{i,j}\frac{\partial^2\rho_{p,1}^k}{\partial x_i\partial x_j} + <\nabla\rho_{p,1}^k, b(t,x,\rho_{p,1})> +\hat{b}^k(t,x,\rho_{p,1}) = 0, \text{ in } (0,T)\times\Omega_p, \\
\rho_{p,1}^k(.,.,a_p) = \rho_{p-1,0}^k(.,.,a_p), \text{ on } (0,T)\times(-l,l)^{n-1}, \\
\rho_{p,1}^k(.,.,b_p) = \rho_{p+1,0}^k(.,.,b_p), \text{ on } (0,T)\times(-l,l)^{n-1}, \\
\rho_{p,1}^k(t,x) = g^k(t,x), \text{ on } (0,T)\times(\partial\mathcal{O}_l\backslash((-l,l)^{n-1}\times(\{a_p\}\cup\{b_p\}))), \\
\rho_{p,1}^k(0,x) = g^k(0,x), \text{ on } (-l,l)^{n-1}\times(a_p,b_p).
\end{cases}
\tag{3.7}
$$

Using $(A5)$, we deduce from (3.7) that

$$
-\frac{\partial\rho_{p,1}^k}{\partial t} + \sum_{i,j=1}^n a_{i,j}\frac{\partial^2\rho_{p,1}^k}{\partial x_i\partial x_j} + <\nabla\rho_{p,1}^k, b(t,x,\rho_{p,1})> +c(t,x)\rho_{p,1}^k = 0,
$$

where

$$
c(t,x) = \begin{cases}
\frac{\hat{b}^k(t,x,\rho_{p,1}) - \hat{b}^k(t,x,\rho_{p,1}^1,\ldots,\rho_{p,1}^{k-1},0,\rho_{p,1}^{k+1},\ldots,\rho_{p,1}^m)}{\rho_{p,1}^k} & \text{if } \rho_{p,1}^k \neq 0, \\
0 & \text{otherwise .}
\end{cases}
$$

Since $c(t,x)$ is negative, by applying the maximum principle (see [12]) to this equation, we can see that the maximum and minimum of $\rho_{p,1}^k$ can be obtained on the boundaries, for all k in $\{1,\ldots,m\}$. This means that $||\rho_{p,1}||_{C([0,T]\times\Omega_p)}$ is bounded by M_0, and then M_0 is also an upper bound of $||\theta_{p,1}^l||_{C([0,T]\times\Omega_p)}$.

Suppose that up to step q_0, the unique classical solution $\theta_{p,q_0}^l(t,x)$ exists, $\theta_{p,q_0}^l(t,x)$, $\frac{\partial}{\partial t}\theta_{p,q_0}^l(t,x)$, $\nabla\theta_{p,q_0}^l(t,x)$, $\Delta\theta_{p,q_0}^l(t,x)$ are bounded, and for all p in $\{1,\ldots,I\}$, $||\theta_{p,q_0}^l||_{C([0,T]\times\Omega_p)}$ is bounded by M_0. We will show that the conclusion is still correct for the step q_0+1. The existence and uniqueness of θ_{p,q_0+1}^l can be infered by using the same argument as in step #1 and Theorem

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2.1. Now, we consider the following equation, for k in $\{1, \ldots, m\}$,

$$
\begin{cases}
-\frac{\partial \rho_{p,q_0+1}^k}{\partial t} + \sum_{i,j=1}^n a_{i,j} \frac{\partial^2 \rho_{p,q_0+1}^k}{\partial x_i \partial x_j} + <\nabla \rho_{p,q_0+1}^k, b(t,x,\rho_{p,q_0+1})> + \\
\qquad\qquad + \hat{b}^k(t,x,\rho_{p,q_0+1}) = 0, \text{ in } (0,T) \times \Omega_p, \\
\rho_{p,q_0+1}^k(.,.,a_p) = \rho_{p-1,q_0}^k(.,.,a_p), \text{ on } (0,T) \times (-l,l)^{n-1}, \\
\rho_{p,q_0+1}^k(.,.,b_p) = \rho_{p+1,q_0}^k(.,.,b_p), \text{ on } (0,T) \times (-l,l)^{n-1}, \\
\rho_{p,q_0+1}^k(t,x) = g^k(t,x), \text{ on } (0,T) \times (\partial \mathcal{O}_l \backslash ((0,T) \times (-l,l)^{n-1} \times (\{a_p\} \cup \{b_p\}))), \\
\rho_{p,q_0+1}^k(0,x) = g^k(0,x), \text{ on } (-l,l)^{n-1} \times (a_p,b_p).
\end{cases}
$$
$$(3.8)$$

Again, by a maximum principle argument applied to Equation (3.8), we can see that the maximum and mininum of ρ_{p,q_0+1}^k can only be obtained on the boundaries, for all k in $\{1, \ldots, m\}$. However, we know that $||\theta_{p,q_0+1}^l||_{C([0,T]\times\Omega_p)}$ is bounded by M_0 for all p in $\{1, \ldots, I\}$, from the definition of M_0. We then deduce that $||\theta_{p,q_0+1}^l||_{C([0,T]\times\Omega_p)}$ is bounded by M_0.

We then conclude that at each step #q in each subdomain #p, there exists a unique classical solution $\theta_{p,q}(t,x)$ for (3.2) and the sequence $\{\theta_{p,q}^l\}_{p \in \overline{1,I}; q \in \mathbb{N}}$ is uniformly bounded with respect to p and q in $C((0,T) \times \mathcal{O}_l)$ by M_0. This concludes the proof.

3.3   Proof of Theorem 3.2

We divide the proof into 2 steps
Step 1: An exponential decay estimate
   Setting $e_{p,q}$ to be $\theta_{p,q}^l - \theta^l$, we deduce the system

$$
\begin{cases}
\frac{\partial e_{p,q}}{\partial t} + \sum_{i,j=1}^n a_{i,j}(t,x,\theta_{p,q}^l) \frac{\partial^2 e_{p,q}}{\partial x_i \partial x_j} + <\nabla e_{p,q}, b(t,x,\theta_{p,q}^l)> +c(t,x,\theta_{p,q}^l,\theta^l) = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T) \times \Omega_p, \\
e_{p,q}(.,.,a_p) = e_{p-1,q-1}(.,.,a_p), \text{ on } (0,T) \times (-l,l)^{n-1}, \\
e_{p,q}(.,.,b_p) = e_{p+1,q-1}(.,.,b_p), \text{ on } (0,T) \times (-l,l)^{n-1}, \\
e_{p,q}(t,x) = 0, \text{ on } (0,T) \times (\partial \mathcal{O}_l \backslash ((0,T) \times (-l,l)^{n-1} \times (\{a_p\} \cup \{b_p\}))), \\
e_{p,q}(T,x) = 0, \text{ on } (-l,l)^{n-1} \times (a_p,b_p),
\end{cases}
$$
$$(3.9)$$

where

$$c(t, x, \theta^l_{p,q}, \theta^l) = \left[\sum_{i,j=1}^{n} [a_{i,j}(t, x, \theta^l_{p,q}) - a_{i,j}(t, x, \theta^l)] \frac{\partial^2 \theta^l}{\partial x_i \partial x_j} \right]$$
$$+ < \nabla \theta^l, [b(t, x, \theta^l_{p,q}) - b(t, x, \theta^l)] > +[\hat{b}(t, x, \theta^l_{p,q}) - \hat{b}(t, x, \theta^l)].$$

Now, defining $\epsilon_{p,q}(t, x, y, z)$ to be $e_{p,q}(T - t, x, y, z)$, we change the system into

$$\begin{cases} \frac{\partial \epsilon_{p,q}}{\partial t} - \sum_{i,j=1}^{n} a_{i,j}(t, x, \theta^l_{p,q}) \frac{\partial^2 \epsilon_{p,q}}{\partial x_i \partial x_j} - < \nabla \epsilon_{p,q}, b(t, x, \theta^l_{p,q}) > -c(t, x, \theta^l_{p,q}, \theta^l) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0, T) \times \Omega_p, \\ \epsilon_{p,q}(., ., a_p) = \epsilon_{p-1,q-1}(., ., a_p), \text{ on } (0, T) \times (-l, l)^{n-1}, \\ \epsilon_{p,q}(., ., b_p) = \epsilon_{p+1,q-1}(., ., b_p), \text{ on } (0, T) \times (-l, l)^{n-1}, \\ \epsilon_{p,q}(t, x) = 0, \text{ on } (0, T) \times (\partial \mathcal{O}_l \backslash ((-l, l)^{n-1} \times (\{a_p\} \cup \{b_p\}))), \\ \epsilon_{p,q}(0, x) = 0, \text{ on } (-l, l)^{n-1} \times (a_p, b_p). \end{cases}$$
$$(3.10)$$

We define

$$\Phi_{p,q}(t, x) = \sum_{k=1}^{m} (\epsilon^k_{p,q})^2 \exp(\beta(x_n - \omega) - \gamma t),$$

where β, ω, γ will be fixed below, and consider the following parabolic operator

$$\mathfrak{L}(\Phi) = \frac{\partial \Phi}{\partial t} - < \nabla \Phi, b(t, x, \theta^l) > - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 \Phi}{\partial x_i x_j} + \sum_{i=1}^{n} \beta a_{i,n} \frac{\partial \Phi}{\partial x_i}. \quad (3.11)$$

A direct computation gives

$$\mathfrak{L}(\Phi_{p,q}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.12)$$
$$= \sum_{k=1}^{m} (-\gamma - \beta b^n(t, x, \theta^l_{p,q}) + \beta a_{n,n} - \beta^2 a_{n,n})(\epsilon^k_{p,q})^2 \exp(\beta(x_n - \omega) - \gamma)$$
$$+ 2\epsilon^k_{p,q} \left[- < \nabla \epsilon^k_{p,q}, b(t, x, \theta^l_{p,q}) > - \sum_{i,j=1}^{n} a_{i,j} \epsilon^k_{p,q} + \frac{\partial \epsilon^k_{p,q}}{\partial t} \right]$$

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$$- \sum_{i,j=1}^{n} 2a_{i,j}(t,x,\theta_{p,q}^l) \frac{\partial \epsilon_{p,q}^k}{\partial x_i} \frac{\partial \epsilon_{p,q}^k}{\partial x_j}$$

$$\leq \sum_{k=1}^{m} \left[(-\gamma - \beta b^n(t,x,\theta_{p,q}^l) - \beta^2 a_{n,n})(\epsilon_{p,q}^k)^2 \exp(\beta(x_n - \omega) - \gamma) \right.$$
$$\left. + 2\epsilon_{p,q}^k c^k(t,x,\theta_{p,q}^l,\theta^l) \right],$$

where c^k is the k-th component of the vector c.
We consider the following term of (3.12)

$$A = \sum_{k=1}^{m} \{ (-\gamma - \beta b^n(t,x,\theta_{p,q}^l) + \beta a_{n,n} - \beta^2 a_{n,n})(\epsilon_{p,q}^k)^2 + 2\epsilon_{p,q}^k c^k(t,x,\theta_{p,q}^l,\theta^l) \}$$

$$= \sum_{k=1}^{m} \{ (-\gamma - \beta b^n(t,x,\theta_{p,q}^l) + \beta a_{n,n} - \beta^2 a_{n,n})(\epsilon_{p,q}^k)^2$$

$$+ 2\epsilon_{p,q}^k [\sum_{i,j=1}^{n} [a_{i,j}(t,x,\theta_{p,q}^l) - a_{i,j}(t,x,\theta^l)] \frac{\partial^2 \theta^{l,k}}{\partial x_i \partial x_j}$$

$$+ <\nabla \theta^{l,k}, [b(t,x,\theta_{p,q}^l) - b(t,x,\theta^l)] > + [\hat{b}(t,x,\theta_{p,q}^l) - \hat{b}(t,x,\theta^l)]] \}$$

$$\leq \{ \sum_{k=1}^{m} (-\gamma - \beta b^n(t,x,\theta_{p,q}^l) + \beta a_{n,n} - \beta^2 a_{n,n})(\epsilon_{p,q}^k)^2$$

$$+ N_1(\epsilon_{p,q}^k)^2 ||\Delta \theta^l||_{C(\mathbb{R}^n)} + N_2(\epsilon_{p,q}^k)^2 ||\nabla \theta^l||_{C(\mathbb{R}^n)} + N_3(\epsilon_{p,q}^k)^2 \}$$

$$\leq \sum_{k=1}^{m} \{ (-\gamma - \beta b^n(t,x,\theta_{p,q}^l) + \beta a_{n,n} - \beta^2 a_{n,n})(\epsilon_{p,q}^k)^2 + N_4(\epsilon_{p,q}^k)^2 \}, \quad (3.13)$$

where N_1, N_2, N_3, N_4 are constants depending only on the coefficients of the system and the bound M_0 of $\theta_{p,q}^l$ and g in $C(\mathbb{R}^n)$. Since all solutions of the subproblems $\{\theta_{p,q}^l\}$ are uniformly bounded, A is negative when γ is large enough and β is suitable chosen. This implies that $\mathfrak{L}(\Phi_{p,q})$ is negative. According to the maximum principle, the maximum of $\Phi_{p,q}$ can only be attained on the boundary of the domain. Which means that the maximum of

$$\sum_{k=1}^{m} (e_{p,q}^k)^2 \exp(\beta(x_n - \omega) - \gamma t)$$

can only be attained on $\{0\} \times \mathbb{R}^{n-1} \times [a_p, b_p]$, on $(0,T) \times (\partial \mathcal{O}_l \backslash ((-l,l)^{n-1} \times (\{a_p\} \cup \{b_p\})))$ or on $([0,T] \times \mathbb{R}^{n-1} \times \{a_p\}) \cup ([0,T] \times \mathbb{R}^{n-1} \times \{b_p\})$.

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Since $\Phi_{p,q}(t,x)$ is equal to 0 on $\{0\}\times\mathbb{R}^{n-1}\times[a_p,b_p]$ and on $(0,T)\times(\partial\mathcal{O}_l\backslash((-l,l)^{n-1}\times(\{a_p\}\cup\{b_p\})))$, we have the following cases:
If $1 < p < I$,

$$\sum_{k=1}^{m}(e_{p,q}^k(t,x))^2\exp(\beta(x_n-\omega)-\gamma t) \qquad (3.14)$$

$$\leq \max\left\{\max_{(t,x)\in[0,T]\times[-l,l]^{n-1}\times\{a_p\}}\sum_{k=1}^{m}(e_{p,q}^k(t,x))^2\exp(\beta(a_p-\omega)-\gamma t),\right.$$

$$\left.\max_{(t,x)\in[0,T]\times[-l,l]^{n-1}\times\{b_p\}}\sum_{k=1}^{m}(e_{p,q}^k(t,x))^2\exp(\beta(b_p-\omega)-\gamma t)\right\}.$$

If $p = 1$

$$\sum_{k=1}^{m}(e_{1,q}^k(t,x))^2\exp(\beta(x_n-\omega)-\gamma t) \qquad (3.15)$$

$$\leq \max_{(t,x)\in[0,T]\times[-l,l]^{n-1}\times\{b_1\}}\sum_{k=1}^{m}(e_{1,q}^k(t,x))^2\exp(\beta(b_1-\omega)-\gamma t).$$

If $p = I$

$$\sum_{k=1}^{m}(e_{I,q}^k(t,x))^2\exp(\beta(x_n-\omega)-\gamma t) \qquad (3.16)$$

$$\leq \max_{(t,x)\in[0,T]\times[-l,l]^{n-1}\times\{a_I\}}\sum_{k=1}^{m}(e_{I,q}^k(t,x))^2\exp(\beta(a_I-\omega)-\gamma t).$$

Step 2: Proof of the Convergence
   Step 2.1: Estimate of the right boudaries of the sub-domains.
   For x in $[-l,l]^n$, we denote x by (X,x_n), where $X\in[-l,l]^{n-1}$ and $x_n\in[-l,l]$. Moreover, we define

$$E_q = \max_{p\in\{1,\dots,I\}}\left\{\max_{(t,x)\in[0,T]\times[-l,l]^n}\sum_{k=1}^{m}(e_{p,q}^k(t,x))^2\exp(-\gamma t)\right\}.$$

Consider the I-th domain, at the q-th step, we can see that (3.16) infers

$$\sum_{k=1}^{m}(e_{I,q}^k(t,X,x_n))^2\exp(\beta(x_n-a_I)-\gamma t)$$

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$$\leq \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{I,q}^{k}(t,X,a_I))^2 \exp(-\gamma t),$$

where ω is replaced by a_I.

Replacing x_n by b_{I-1} in the previous inequality, we obtain

$$\sum_{k=1}^{m} (e_{I,q}^{k}(t,X,b_{I-1}))^2 \exp(\beta(b_{I-1}-a_I)-\gamma t)$$

$$\leq \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{I,q}^{k}(t,X,a_I))^2 \exp(-\gamma t).$$

Since $e_{I,q}^{k}(t,X,b_{I-1})$ is equal to $e_{I-1,q+1}^{k}(t,X,b_{I-1})$, then

$$\sum_{k=1}^{m} (e_{I-1,q+1}^{k}(t,X,b_{I-1}))^2 \exp(\beta(b_{I-1}-a_I)-\gamma t)$$

$$\leq \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{I,q}^{k}(t,X,a_I))^2 \exp(-\gamma t).$$

We define β_1 to be $\sqrt{\frac{\gamma}{2}}$ and let β in this case be β_1; then if we choose γ large, $\gamma - \beta^2$ is large, the inequality becomes

$$\sum_{k=1}^{m} (e_{I-1,q+1}^{k}(t,X,b_{I-1}))^2 \exp(-\gamma t)$$

$$\leq \exp(-\beta_1 S_{I-1}) \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{I,q}^{k}(t,X,a_I))^2 \exp(-\gamma t).$$

We deduce that

$$\sum_{k=1}^{m} (e_{I-1,q+1}^{k}(t,X,b_{I-1}))^2 \exp(-\gamma t) \leq \exp(-\beta_1 S_{I-1}) E_q. \qquad (3.17)$$

Moreover, on the $(I-1)$-th domain, at the $(q+1)$-th step, (3.14) leads to

$$\sum_{k=1}^{m} (e_{I-1,q+1}^{k}(t,X,x_n))^2 \exp(\beta(x_n-a_{I-1})-\gamma t)$$

$$\leq \quad \max\left\{\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m}(e_{I-1,q+1}^{k}(t,X,a_{I-1}))^2 \exp(-\gamma t),\right.$$

$$\left.\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m}(e_{I-1,q+1}^{k}(t,X,b_{I-1}))^2 \exp(\beta(b_{I-1}-a_{I-1})-\gamma t)\right\},$$

where ω is replaced by a_{I-1}.

Since $e_{I-1,q+1}^{k}(t,X,b_{I-2})$ is equal to $e_{I-2,q+2}^{k}(t,X,b_{I-2})$, then

$$\sum_{k=1}^{m}(e_{I-2,q+2}^{k}(t,X,b_{I-2}))^2 \exp(\beta(b_{I-2}-a_{I-1})-\gamma t)$$

$$\leq \quad \max\left\{\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m}(e_{I-1,q+1}^{k}(t,X,a_{I-1}))^2 \exp(-\gamma t),\right.$$

$$\left.\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m}(e_{I-1,q+1}^{k}(t,X,b_{I-1}))^2 \exp(\beta L_{I-1}-\gamma t)\right\};$$

thus

$$\sum_{k=1}^{m}(e_{I-2,q+2}^{k}(t,X,b_{I-2}))^2 \exp(\beta S_{I-2}-\gamma t)$$

$$\leq \quad \max\left\{\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m}(e_{I-1,q+1}^{k}(t,X,a_{I-1}))^2 \exp(-\gamma t),\right.$$

$$\left.\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m}(e_{I-1,q+1}^{k}(t,X,b_{I-1}))^2 \exp(\beta L_{I-1}-\gamma t)\right\}.$$

Combining this with (3.17) and the fact that

$$\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m}(e_{I-1,q+1}^{k}(t,X,a_{I-1}))^2 \exp(-\gamma t) \leq E_{q+1},$$

we obtain that

$$\sum_{k=1}^{m}(e_{I-2,q+2}^{k}(t,X,b_{I-2}))^2 \exp(\beta S_{I-2}-\gamma t) \leq \max\{E_q \exp(\beta L_{I-1}-\beta_1 S_{I-1}), E_{q+1}\}.$$

Thus

$$\sum_{k=1}^{m}(e_{I-2,q+2}^{k}(t,X,b_{I-2}))^{2}\exp(-\gamma t) \leq \max\{E_q\exp(\beta(L_{I-1}-S_{I-2})-\beta_1 S_{I-1}),$$

$$E_{q+1}\exp(-\beta S_{I-2})\}.$$

Defining β_2 to be $\beta_1\frac{S_{I-1}}{L_{I-1}}$ and choosing β to be β_2 such that

$$\beta_2(-L_{I-1}+S_{I-2})+\beta_1 S_{I-1}=\beta_2 S_{I-2},$$

we infer

$$\sum_{k=1}^{m}(e_{I-2,q+2}^{k}(t,X,b_{I-2}))^{2}\exp(-\gamma t) \leq \max\{E_k,E_{k+1}\}\exp(-\beta_2 S_{I-2}). \quad (3.18)$$

Using the same techniques as the ones that we use to achive (3.17) and (3.18), we can prove that

$$\sum_{k=1}^{m}(e_{I-j,q+j}^{k}(t,X,b_{I-j}))^{2}\exp(-\gamma t) \leq \max\{E_k,\ldots,E_{k+j-1}\}\exp(-\beta_j S_{I-j}),$$

$$(3.19)$$

where

$$\beta_j = \beta_1\frac{S_{I-1}}{L_{I-1}}\ldots\frac{S_{I-j+1}}{L_{I-j+1}}, \quad j \in \{2,\ldots,I-1\}.$$

Step 2.2: Estimate of the left boudaries of the sub-domains
Consider the 1-th domain, at the k-th step. Then (3.15) infers that

$$\sum_{k=1}^{m}(e_{1,q}^{k}(t,X,x_n))^{2}\exp(\beta(x_n-b_1)-\gamma t)$$

$$\leq \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}(e_{1,q}^{k}(t,X,b_1))^{2}\exp(-\gamma t),$$

we notice here that ω is replaced by b_1.
Replace x_n by a_2, we obtain that

$$\sum_{k=1}^{m}(e_{1,q}^{k}(t,X,a_2))^{2}\exp(\beta(a_2-b_1)-\gamma t)$$

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$$\leq \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{1,q}^k(t,X,b_1))^2 \exp(-\gamma t),$$

Since $e_{1,q}^k(t,X,a_2)$ is equal to $e_{2,q+1}^k(t,X,a_2)$,

$$\sum_{k=1}^{m} (e_{2,q+1}^k(t,X,a_2))^2 \exp(-\gamma t)$$

$$\leq \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{1,q}^k(t,X,b_1))^2 \exp(-\gamma t)\exp(\beta S_1),$$

We define β_1' to be $-\sqrt{\frac{\gamma}{2}}$ and let β be β_1' in this case. If we choose γ large, $\gamma - \beta^2$ is large. The inequality becomes

$$\sum_{k=1}^{m} (e_{2,q+1}^k(t,X,a_2))^2 \exp(-\gamma t)$$

$$\leq \exp(-\beta_1' S_1) \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{1,q}^k(t,X,b_1))^2 \exp(-\gamma t).$$

We deduce

$$\sum_{k=1}^{m} (e_{2,q+1}^k(t,X,a_2))^2 \exp(-\gamma t) \leq \exp(-\beta_1' S_1) E_q. \qquad (3.20)$$

Moreover, on the 2-th domain, at the $(q+1)$-th step, (3.14) leads to

$$\sum_{k=1}^{m} (e_{2,q+1}^k(t,X,x_n))^2 \exp(\beta(x_n - a_2) - \gamma t)$$

$$\leq \max \left\{ \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{2,q+1}^k(t,X,a_2))^2 \exp(-\gamma t), \right.$$

$$\left. \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{2,q+1}^k(t,X,b_2))^2 \exp(\beta(b_2 - a_2) - \gamma t) \right\},$$

notice that ω is replaced by a_2.
Since $e_{2,q+1}^k(t,X,a_3)$ is equal to $e_{3,q+2}^k(t,X,a_3)$, then

$$\sum_{k=1}^{m} (e_{3,q+2}^k(t,X,a_3))^2 \exp(\beta(a_3 - a_2) - \gamma t)$$

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$$\leq \quad \max \left\{ \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{2,q+1}^{k}(t, X, a_2))^2 \exp(-\gamma t) \, , \right.$$

$$\left. \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{2,q+1}^{k}(t, X, b_2))^2 \exp(\beta(b_2 - a_2) - \gamma t) \right\}.$$

Hence

$$\sum_{k=1}^{m} (e_{3,q+2}^{k}(t, X, a_3))^2 \exp(-\gamma t)$$

$$\leq \quad \exp(-\beta(L_2 - S_2)) \max \left\{ \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{2,q+1}^{k}(t, X, a_2))^2 \exp(-\gamma t), \right.$$

$$\left. \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}} \sum_{k=1}^{m} (e_{2,q+1}^{k}(t, X, b_2))^2 \exp(\beta L_2 - \gamma t) \right\}.$$

Combining this inequality, (3.20) and the fact that

$$\sum_{k=1}^{m} (e_{2,q+1}^{k}(t, X, b_2))^2 \exp(-\gamma t) \leq E_{q+1},$$

we deduce

$$\sum_{k=1}^{m} (e_{3,q+2}^{k}(t, X, a_3))^2 \exp(-\gamma t)$$

$$\leq \quad \exp(-\beta(L_2 - S_2)) \max\{\exp(-\beta_1' S_1) E_q, \exp(\beta L_2) E_{q+1}\},$$

Define β_2' to be $\beta_1' \frac{S_1}{L_2}$ and choose β to be $-\beta_2'$, we infer that

$$-\beta(L_2 - S_2) - \beta_1' S_1 = \beta S_2.$$

This implies

$$\sum_{k=1}^{m} (e_{3,q+2}^{k}(t, X, a_3))^2 \exp(-\gamma t) \leq \max\{E_k, E_{k+1}\} \exp(-\beta_2' S_2). \qquad (3.21)$$

Using the same techniques as the ones that we use to achive (3.20) and (3.21), we can prove that

$$\sum_{k=1}^{m} (e_{j,q+j-1}^{k}(t, X, a_j))^2 \exp(-\gamma t) \leq \max\{E_k, \ldots, E_{k+j-2}\} \exp(-\beta_{j-1}' S_{j-1}),$$

$$(3.22)$$

where

$$\beta'_j = \beta'_1 \frac{S_1}{L_2} \dots \frac{S_{j-1}}{L_j}, \quad j \in \{2, \dots, I-1\}.$$

Step 2.3: Convergence result
Setting

$$\bar{\epsilon} = \sqrt{\frac{\gamma}{2}} \frac{S_1 \dots S_{I-1}}{L_2 \dots L_{I-1}},$$

and

$$\bar{E}_k = \max_{j \in \{0, \dots, I-1\}} \{E_{k+j}\},$$

we infer from (3.19) and (3.22) that

$$\bar{E}_{k+1} \leq \bar{E}_k \exp(-\bar{\epsilon}), \forall k \in \mathbb{N}.$$

Therefore

$$\bar{E}_n \leq \bar{E}_0 \exp(-n\bar{\epsilon}), \forall n \in \mathbb{N},$$

Hence E_k tends to 0 as k tends to infinity. Which gives that

$$\lim_{q \to \infty} \max_{p=\overline{1,I}} \sum_{k=1}^{m} ||e_{p,q}^k||_{C([0,T] \times \mathbb{R}^{n-1} \times [a_p, b_p])} = 0,$$

and this concludes the proof.

3.4 Proof of Theorem 3.3

We start proving that

$$\lim_{l \to \infty} \lim_{q \to \infty} \int_0^T E(|X_t^{q,l} - X_t|^2) dt = 0.$$

Subtracting (2.6) and (3.5), we get

$$
\begin{aligned}
X_t - X_t^{q,l} &= \int_0^t [b(s, X_s, \theta(s, X_s)) - b(s, X_s^{q,l}, \theta^{q,l}(s, X^{q,l}))] ds \qquad (3.23) \\
&\quad + \int_0^t [\sigma((s, X_s, \theta(s, X_s)) - \sigma(s, X_s^{q,l}, \theta^{q,l}(s, X_s^{q,l}))] dW_s,
\end{aligned}
$$

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which leads to

$$|X_t - X_t^{q,l}|^2 \leq 2\left(\int_0^t [b(s, X_s, \theta(s, X_s)) - b(s, X_s^{q,l}, \theta^{q,l}(s, X_s^{q,l}))]ds\right)^2 \quad (3.24)$$

$$+2\left(\int_0^t [\sigma(s, X_s, \theta(s, X_s)) - \sigma(s, X_s^{q,l}, \theta^{q,l}(s, X_s^{q,l}))]dW_s\right)^2.$$

A simple calculation gives

$$E(|X_t - X_t^{q,l}|^2) \leq N_5 E \int_0^t |b(s, X_s, \theta(s, X_s)) - b(s, X_s^{q,l}, \theta^{q,l}(s, X_s^{q,l}))|^2 ds \quad (3.25)$$

$$+N_5 E \int_0^t |\sigma(s, X_s, \theta(s, X_s)) - \sigma(s, X_s^{q,l}, \theta^{q,l}(s, X_s^{q,l}))|^2 ds$$

$$\leq N_6 E \int_0^t [|X_s - X_s^{q,l}|^2 + |\theta(s, X_s) - \theta^{q,l}(s, X_s^{q,l})|^2] ds$$

$$\leq N_6 E \int_0^t |X_s - X_s^{q,l}|^2 ds$$

$$+ N_7 E \int_0^t [|\theta(s, X_s^{q,l}) - \theta^{q,l}(s, X_s^{q,l})|^2 + |\theta(s, X_s) - \theta(s, X_s^{q,l})|^2] ds$$

$$\leq N_8 E \int_0^t [|X_s - X_s^{q,l}|^2 + |\theta(s, X_s^{q,l}) - \theta^{q,l}(s, X_s^{q,l})|^2] ds,$$

where N_5, N_6, N_7, N_8 are positive constants.

Since $\{\theta^{q,l}\}$ converges uniformly to θ^l, and $\{\theta^l\}$ converges uniformly to θ, then for all positive number ϵ, there exists $Q(\epsilon)$ such that

$$|\theta(s, X_s^{q,l}) - \theta^{q,l}(s, X_s^{q,l})| < \sqrt{\epsilon}, \forall q, l > Q(\epsilon).$$

Inequality (3.25) leads to

$$E(|X_t - X_t^{q,l}|^2) \leq N_8 E \int_0^t [|X_s - X_s^{q,l}|^2 + \epsilon] ds, \quad \forall q, l > Q(\epsilon). \quad (3.26)$$

Now, setting

$$H(t) = \int_0^t E(|X_s - X_s^{q,l}|^2) ds,$$

we obtain

$$H'(t) \leq N_9 H(t) + N_9 \epsilon, \quad (3.27)$$

where N_9 is a positive constant. Therefore

$$H'(t)\exp(-N_9 t) - N_9 H(t)\exp(-N_9 t) - N_9\epsilon\exp(-N_9 t) \le 0.$$

This implies

$$(H(t)\exp(-N_9 t) + \epsilon\exp(-N_9 t))' \le 0,$$

and this inequality then leads to

$$H(t)\exp(-N_9 t) + \epsilon\exp(-N_9 t) \le \epsilon.$$

Consequently,

$$H(t) \le \epsilon(\exp(N_9 T) - 1)$$

for q, l greater than $Q(\epsilon)$, which leads to

$$\lim_{l\to\infty}\lim_{q\to\infty}\int_0^T E(|X_t - X_t^{q,l}|^2)dt = 0.$$

We next prove that

$$\lim_{l\to\infty}\lim_{q\to\infty}\int_0^T E(|Y_t^{q,l} - Y_t|^2)dt = 0.$$

The fact that

$$\lim_{l\to\infty}\lim_{q\to\infty}\int_0^T E(|X_t - X_t^{q,l}|^2)dt = 0$$

implies that $X_t^{q,l}$ converges to X_t almost everywhere. From which, we can infer that $\theta^{q,l}(t, X_t^{q,l})$ converges to $\theta(t, X_t)$ almost everywhere. Since $\{\theta^{q,l}(t, X_t^{q,l})\}$ is bounded, the Lebesgue Dominated Convergence Theorem gives that

$$\lim_{l\to\infty}\lim_{q\to\infty} E\int_t^T |Y_s^{q,l} - Y_s|^2 ds = 0,$$

and then

$$\lim_{l\to\infty}\lim_{q\to\infty}\int_0^T E(|Y_t^{q,l} - Y_t|^2)dt = 0.$$

Now, we prove that

$$\lim_{l\to\infty}\lim_{q\to\infty} E\int_0^T |Z_t^{q,l} - Z_t|^{\frac{2}{\alpha}}dt = 0.$$

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We get the following equation from the definition of Y_t and $Y_t^{q,l}$,

$$Y_t - Y_t^{q,l} = \int_t^T [\hat{b}(s, X_s, Y_s) - \hat{b}(s, X_s^{q,l}, Y_s^{q,l})]ds \qquad (3.28)$$

$$+ \int_t^T [\hat{\sigma}(s, X_s, Y_s, Z_s) - \hat{\sigma}(s, X_s^{q,l}, Y_s^{q,l}, Z_s^{q,l})]dW_s.$$

This implies that

$$E \int_t^T [\hat{\sigma}(s, X_s, Y_s, Z_s) - \hat{\sigma}(s, X_s, Y_s, Z_s^{q,l})]^2 ds \qquad (3.29)$$

$$\leq N_{10} \left(E(|Y_t - Y_t^{q,l}|^2) + E \int_t^T [\hat{b}(s, X_s, Y_s) - \hat{b}(s, X_s^{q,l}, Y_s^{q,l})]^2 ds \right)$$

$$\leq N_{11} \left(E(|Y_t - Y_t^{q,l}|^2) + E \int_t^T [|X_s - X_s^{q,l}|^2 + |Y_s - Y_s^{q,l}|^2] ds \right),$$

where N_{10}, N_{11} are positive constants.
Applying Condition $(A3)$ to the inequality (3.29), we get

$$E \int_t^T |Z_s - Z_s^{q,l}|^{\frac{2}{\alpha}} ds \qquad (3.30)$$

$$\leq N_{12} \left(E(|Y_t - Y_t^{q,l}|^2) + E \int_t^T [|X_s - X_s^{q,l}|^2 + |Y_s - Y_s^{q,l}|^2] ds \right),$$

where N_{12} is a positive constant.
Due to the fact that

$$\lim_{l \to \infty} \lim_{q \to \infty} \int_0^T E(|X_t - X_t^{q,l}|^2) dt = 0,$$

and

$$\lim_{l \to \infty} \lim_{q \to \infty} E \int_0^T |Y_t - Y_t^{q,l}|^2 dt = 0,$$

we can see that the RHS of (3.30) tends to 0 as q and l tend to infinity. Hence $E \int_0^T |Z_t - Z_t^{q,l}|^{\frac{2}{\alpha}} dt$ tends to 0 as q and l tend to infinity. This concludes the proof.

4 Conclusion

We have introduced a new Domain Decomposition Method for a system of SDEs. The method has been studied theoretically and proved to be well-posed and stable. We have also proposed a new technique to prove the convergence of Domain Decomposition Methods for systems of nonlinear parabolic equations in n-dimension. The method has the potential to be used to prove the convergence of Domain Decomposition Methods for many kinds of nonlinear problems.

References

[1] Fabio Antonelli. Backward-forward stochastic differential equations. *Ann. Appl. Probab.*, 3(3):777–793, 1993.

[2] Lori Badea. On the Schwarz alternating method with more than two subdomains for nonlinear monotone problems. *SIAM J. Numer. Anal.*, 28(1):179–204, 1991.

[3] Christian Bender and Jianfeng Zhang. Time discretization and Markovian iteration for coupled FBSDEs. *Ann. Appl. Probab.*, 18(1):143–177, 2008.

[4] Kevin Burrage, Carolyn Dyke, and Bert Pohl. On the performance of parallel waveform relaxations for differential systems. *Appl. Numer. Math.*, 20:39–55, 1996.

[5] Jakša Cvitanić and Jianfeng Zhang. The steepest descent method for forward-backward SDEs. *Electron. J. Probab.*, 10:1468–1495 (electronic), 2005.

[6] F. Delarue and G. Guatteri. Weak existence and uniqueness for forward-backward SDEs. *Stochastic Process. Appl.*, 116(12):1712–1742, 2006.

[7] François Delarue. On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case. *Stochastic Process. Appl.*, 99(2):209–286, 2002.

[8] François Delarue and Stéphane Menozzi. A forward-backward stochastic algorithm for quasi-linear PDEs. *Ann. Appl. Probab.*, 16(1):140–184, 2006.

[9] Stephane Descombes, Victorita Dolean, and Martin J. Gander. Schwarz waveform relaxation methods for systems of semi-linear reaction-diffusion equations. In *Domain Decomposition Methods*, 2009.

[10] Jim Douglas, Jr., Jin Ma, and Philip Protter. Numerical methods for forward-backward stochastic differential equations. *Ann. Appl. Probab.*, 6(3):940–968, 1996.

[11] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, 7(1):1–71, 1997.

[12] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[13] Martin J. Gander. A waveform relaxation algorithm with overlapping splitting for reaction diffusion equations. *Numer. Linear Algebra Appl.*, 6(2):125–145, 1999. Czech-US Workshop in Iterative Methods and Parallel Computing, Part 2 (Milovy, 1997).

[14] Martin J. Gander and Andrew M. Stuart. Space-time continuous analysis of waveform relaxation for the heat equation. *SIAM J. Sci. Comput.*, 19(6):2014–2031 (electronic), 1998.

[15] Eldar Giladi and Herbert B. Keller. Space time domain decomposition for parabolic problems. *Numerische Mathematik*, 93(2):279–313, 2002.

[16] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.

[17] P.-L. Lions. On the Schwarz alternating method. I. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987)*, pages 1–42. SIAM, Philadelphia, PA, 1988.

[18] P.-L. Lions. On the Schwarz alternating method. II. Stochastic interpretation and order properties. In *Domain decomposition methods (Los Angeles, CA, 1988)*, pages 47–70. SIAM, Philadelphia, PA, 1989.

[19] P.-L. Lions. On the Schwarz alternating method. III. A variant for nonoverlapping subdomains. In *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Houston, TX, 1989)*, pages 202–223. SIAM, Philadelphia, PA, 1990.

[20] S. H. Lui. On Schwarz methods for monotone elliptic PDEs. In *Domain decomposition methods in sciences and engineering (Chiba, 1999)*, pages 55–62 (electronic). DDM.org, Augsburg, 2001.

[21] S. H. Lui. On linear monotone iteration and Schwarz methods for nonlinear elliptic PDEs. *Numer. Math.*, 93(1):109–129, 2002.

[22] Shiu-Hong Lui. On monotone and Schwarz alternating methods for nonlinear elliptic PDEs. *M2AN Math. Model. Numer. Anal.*, 35(1):1–15, 2001.

[23] Jin Ma, Philip Protter, and Jiong Min Yong. Solving forward-backward stochastic differential equations explicitly—a four step scheme. *Probab. Theory Related Fields*, 98(3):339–359, 1994.

[24] Jin Ma, Jie Shen, and Yanhong Zhao. On numerical approximations of forward-backward stochastic differential equations. *SIAM J. Numer. Anal.*, 46(5):2636–2661, 2008.

[25] Jin Ma and Jiongmin Yong. Solvability of forward-backward SDEs and the nodal set of Hamilton-Jacobi-Bellman equations. *Chinese Ann. Math. Ser. B*, 16(3):279–298, 1995. A Chinese summary appears in Chinese Ann. Math. Ser. A **16** (1995), no. 4, 532.

[26] Jin Ma and Jiongmin Yong. *Forward-backward stochastic differential equations and their applications*, volume 1702 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999.

[27] G. N. Milstein and M. V. Tretyakov. Numerical algorithms for forward-backward stochastic differential equations. *SIAM J. Sci. Comput.*, 28(2):561–582 (electronic), 2006.

[28] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, volume 176 of *Lecture Notes in Control and Inform. Sci.*, pages 200–217. Springer, Berlin, 1992.

[29] H. Pham. Pde formulation survey. To appear in Encyclopedia of Quantitative Finance, Wiley, ed. R. Cont.